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Some Optimality Conditions for Chebyshev Expansions

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In this paper we investigate conditions under which approximation to continuous functions on $[-1, 1]$ by series of Chebyshev polynomials is superior to approximation by other ultraspherical orthogonal expansions. In particular we derive conditions on the Chebyshev coefficients which guarantee that the Chebyshev expansion of the corresponding functions converges more rapidly than expansions in Legendre polynomials or Chebyshev polynomials of the second kind.

1. INTRODUCTION

In this paper we shall be concerned with approximating functions, with certain smoothness properties, by expansions in terms of orthogonal polynomials. We shall carry out the approximation on the closed interval $[-1, 1]$ using the supremum norm $\|f\| = \max_{x\in[-1,1]} |f(x)|$, and our most common choice of orthogonal polynomials on this interval will be the ultraspherical polynomials $C_n^{(\alpha)}(x)$ $(\alpha > -\frac{1}{2})$ satisfying

$$\int_{-1}^{1} (1 - t^2)^{\alpha-1/2}\, C_n^{(\alpha)}(t)\, C_m^{(\alpha)}(t)\, dt = 0 \qquad (m \neq n).$$

The Chebyshev polynomials, which are members of this class $(\alpha = 0)$, are widely used in numerical analysis. One of their virtues is that expansions of functions in series of Chebyshev polynomials are thought to converge more rapidly than expansions in series of other orthogonal polynomials. Indeed, Lanczos [5] suggested that the terms of the Chebyshev expansion were asymptotically smaller in maximum absolute value than the corresponding terms of any other ultraspherical expansion. His argument contained a major weakness which was rectified by Handscomb [4]. Let the function f have the two (formal) expansions

$$f(x) = \sum_{n=0}^{\infty} a_n^{(\alpha)} C_n^{(\alpha)}(x) = \sum_{n=0}^{\infty} a_n^{(0)} T_n(x),$$

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where the T_n are the Chebyshev polynomials. Define

$$\| a^{(\alpha)} \|_M = \sum_{n=M}^{\infty} | a_n^{(\alpha)} |.$$

Handscomb [4] shows, among other results, that for $\alpha > 0$

$$\| a^{(\alpha)} \|_M \geqslant \| a^{(0)} \|_M .$$

(The normalization of the polynomials used in the proof is $C_n^{(\alpha)}(1) = 1$, $T_n(1) = 1$). Handscomb concludes his paper with a conjecture that no function exists whose ultraspherical terms are smaller than the Chebyshev terms in the context described above.

A slightly different approach was adopted by Rivlin and Wilson [7]. They showed that if

$$f(x) = \sum_{n=0}^{\infty} a_n^{(\alpha)} C_n(x) \qquad \text{and} \qquad \sum_{n=0}^{\infty} | a_n^{(\alpha)} | < \infty,$$

where $C_n^{(\alpha)}(1) = 1$, then either of the conditions

(i) $a_n^{(\alpha)} \geqslant 0$ for $n > m$,

(ii) $(-1)^n a_n^{(\alpha)} \geqslant 0$ for $n > m$,

is sufficient for

$$\left\| f - \sum_{n=0}^{m} a_n^{(\alpha)} C_n^{(\alpha)} \right\| \leqslant \left\| f - \sum_{n=0}^{m} a_n^{(\alpha_1)} C_n^{(\alpha_1)} \right\|$$

for all $\alpha_1 \geqslant \alpha \geqslant 0$. Several special functions (e.g., $f(x) = e^x$) have $a_n^{(0)} \geqslant 0$ for all $n \geqslant 0$, and the theorem of Rivlin and Wilson can be applied immediately to such functions giving

$$\left\| f - \sum_{n=0}^{m} a_n^{(0)} T_n \right\| \leqslant \left\| f - \sum_{n=0}^{m} a_n^{(\alpha)} C_n^{(\alpha)} \right\|$$

for all $\alpha \geqslant 0$ and $m \geqslant 0$. Their paper also contains results in a similar vein for Jacobi polynomials.

Fox and Parker [3] point out that if the Chebyshev coefficients $a_n^{(0)}$ decrease rapidly, then truncation of the series after M terms will give a very close approximation to the minimax polynomial (the minimax polynomial is the best approximation to the function in the supremum norm by polynomials of degree at most M). This statement rests on the fact that the Chebyshev polynomials have the equioscillation behavior required of a minimax approximation, so that whenever f is a polynomial of degree $M + 1$ or less, then

$$\left\| f - \sum_{n=0}^{M} a_n^{(0)} T_n \right\| = \min_p \| f - p \|,$$

where p is any polynomial of degree M.

The purpose of this paper is to compare the performance of ultraspherical approximations for varying α, using the above property of the Chebyshev polynomials as motivation for showing their superior performance. The approach adopted will be both analytical and computational.

2. General Theory

We begin the analysis in a general setting. Suppose $\{p_n\}$ is a sequence of polynomials on $[-1, 1]$ satisfying

(i) p_n is of strict degree n,

(ii) p_n is even or odd in correspondence with the parity of n,

(iii) $\| p_n \| = 1$.

Property (ii) allows us to write $p_n = \sum_{k=0}^{n} \lambda_k^{(n)} C_k^{(\alpha)}$, where $\lambda_k^{(n)} \in R$, and $\lambda_k^{(n)} = 0$ if $k + n$ is odd. By redefining the sequence $\{p_n\}$ with opposite signs if necessary we can also assume that $\lambda_n^{(n)} > 0$. We will further assume that $\alpha > 0$, and $\| C_n^{(\alpha)} \| = C_n^{(\alpha)}(1) = 1$, and can now derive the following theorem:

THEOREM 2.1. *Let f, belonging to $C[-1, 1]$, have the uniformly convergent expansion $f = \sum_{n=0}^{\infty} b_n p_n$. Then a sufficient condition for*

$$\left\| f - \sum_{n=0}^{M} b_n p_n \right\| \leqslant \left\| f - \sum_{n=0}^{M} a_n(\alpha)\, C_n^{(\alpha)} \right\|$$

for a fixed $M \geqslant 0$ is

$$\sum_{n=M+3}^{\infty} | b_n | \left(1 + \left\| p_n - \sum_{k=0}^{M} \lambda_k^{(n)} C_k^{(\alpha)} \right\| \right)$$

$$\leqslant | b_{M+1} | (\lambda_{M+1}^{(M+1)} - 1) + | b_{M+2} | (\lambda_{M+2}^{(M+2)} - 1).$$

Proof. It will be convenient to recall that the operator $L_M^{\alpha}: C[-1, 1] \to P_M[-1, 1]$ given by

$$L_M^{(\alpha)} f = \sum_{n=0}^{M} a_n(\alpha)\, C_n^{\alpha}$$

is a bounded, linear, idempotent operator. Now

$$f - \sum_{n=0}^{M} b_n p_n = \sum_{n=M+1}^{\infty} b_n p_n = \sum_{n=M+1}^{\infty} b_n \sum_{k=0}^{n} \lambda_k^{(n)} C_k^{(\alpha)},$$

and

$$f - \sum_{n=0}^{M} a_n(\alpha)\, C_n^{(\alpha)} = f - L_M^{(\alpha)} f = f - \sum_{n=0}^{M} b_n p_n - L_M^{(\alpha)} \left(f - \sum_{n=0}^{M} b_n p_n \right)$$

since $L_M{}^{\alpha}$ is idempotent.

Hence

$$\begin{aligned} f - \sum_{n=0}^{M} a_n(\alpha)\, C_n^{(\alpha)} &= \sum_{n=M+1}^{\infty} b_n p_n - \sum_{n=M+1}^{\infty} b_n L_M^{(\alpha)} p_n \\ &= \sum_{n=M+1}^{\infty} b_n \left(p_n - \sum_{k=0}^{M} \lambda_k^{(n)} C_k^{(\alpha)} \right) \\ &= b_{M+1} \left(p_{M+1} - \sum_{k=0}^{M} \lambda_k^{(M+1)} C_k^{(\alpha)} \right) \\ &\quad + b_{M+2} \left(p_{M+2} - \sum_{k=0}^{M} \lambda_k^{(M+2)} C_k^{(\alpha)} \right) \\ &\quad + \sum_{n=M+3}^{\infty} b_n \left(p_n - \sum_{k=0}^{M} \lambda_k^{(n)} C_k^{(\alpha)} \right). \end{aligned}$$

Writing $R_{M+3} = \sum_{r=M+3}^{M} b_n (p_n - \sum_{k=0}^{M} \lambda_k^{(n)} C_k^{(\alpha)})$ and observing that

$$P_{M+i} - \sum_{k=0}^{M} \lambda_k^{(M+i)} C_k^{(\alpha)} = \lambda_{M+i}^{(M+i)} C_{M+i}^{(\alpha)}, \qquad i = 1, 2$$

we have

$$\sum_{n=M+1}^{\infty} a_n(\alpha)\, C_n^{(\alpha)} = b_{M+1} \lambda_{M+1}^{(M+1)} C_{M+1}^{(\alpha)} + b_{M+2} \lambda_{M+2}^{(M+2)} C_{M+2}^{(\alpha)} + R_{M+3}\,.$$

Now for $\| f - \sum_{n=0}^{M} b_n p_n \| \leqslant \| f - \sum_{n=0}^{M} a_n(\alpha)\, C_n^{(\alpha)} \|$ we must have

$$\left\| \sum_{n=M+1}^{\infty} b_n p_n \right\| \leqslant \| b_{M+1} \lambda_{M+1}^{(M+1)} C_{M+1}^{(\alpha)} + b_{M+2} \lambda_{M+2}^{(M+2)} C_{M+2}^{(\alpha)} + R_{M+3} \|.$$

In view of the fact that the $C_n^{(\alpha)}$ also satisfy condition (ii), we may obtain the following inequality which is sufficient for the one above:

$$\left\| \sum_{n=M+1}^{\infty} b_n p_n \right\| \leqslant | b_{M+1} |\, \lambda_{M+1}^{(M+1)} + | b_{M+2} |\, \lambda_{M+2}^{(M+2)} - \| R_{M+3} \|.$$

The sufficient condition in the theorem is obtained by replacing $\| R_{M+3} \|$ by the upper bound

$$\| R_{M+3} \| \leqslant \sum_{n=M+3}^{\infty} | b_n | \left\| p_n - \sum_{k=0}^{M} \lambda_k^{(n)} C_k^{(\alpha)} \right\|,$$

and using

$$\left\| \sum_{n=M+1}^{\infty} b_n p_n \right\| \leqslant | b_{M+1} | + | b_{M+2} | + \sum_{n=M+3}^{\infty} | b_n |.$$

The theorem as such has two significant drawbacks, since two different circumstances can give rise to the theorem failing outright. First, the sum on the left-hand side of the inequality may not converge. Second, even though the left-hand sum may converge, the right-hand side could be negative if $\lambda_{M+1}^{(M+1)}$ or $\lambda_{M+2}^{(M+2)}$ are less than unity. However, suppose we replace the p_n by the Chebyshev polynomials T_n. Then it is well known that of all polynomials of degree n with unit norm on $[-1, 1]$, the Chebyshev polynomial has the largest coefficient of x^n. Hence in this case $\lambda_{M+1}^{(M+1)}, \lambda_{M+2}^{(M+2)} > 1$, and the right-hand side is always nonnegative.

EXAMPLE 1. Take the following definitions for p_n and α;

$p_n = T_n$, the Chebyshev polynomials of the first kind,

$$\alpha = 1, \quad \text{so that } C_n^{(1)}(x) = \frac{U_n(x)}{n+1} = \frac{\sin(n+1)\theta}{(n+1)\sin\theta}, \quad x = \cos\theta.$$

These are the Chebyshev polynomials of the second kind.

In this case we have

$$T_n = \frac{(n+1)}{2} C_n^{(1)} - \frac{(n-1)}{2} C_{n-2}^{(1)}$$

giving the $\lambda_k^{(n)}$ of Theorem 2.1 a particularly simple form. Hence a sufficient condition for

$$\left\| f - \sum_{n=0}^{M} b_n T_n \right\| \leqslant \left\| f - \sum_{n=0}^{M} a_n U_n \right\|$$

is

$$\sum_{n=M+3}^{\infty} | b_n | \leqslant \frac{(M-1)}{4} | b_{M+1} | + \frac{M}{4} | b_{M+2} |.$$

Observe that this condition contains *no* reference to the coefficients in the U_n-expansion, and so a given Chebyshev series can easily be tested numerically to see whether it satisfies the condition.

EXAMPLE II. For our second example, we let $p_n = T_n$ and restrict α to lie in (0, 1). We have the following information about the $\lambda_k^{(n)}$ in this case:

LEMMA 2.2. *Let* $T_n = \sum_{k=0}^{n} \lambda_k^{(n)} C_k^{(\alpha)}$ *for* $0 < \alpha < 1$. *Then*

$$\lambda_k^{(n)} \leqslant \lambda_k^{(n+2)} \leqslant 0 \qquad \textit{for} \quad 0 < k < n.$$

Proof. Details may be found in [6].

These facts help us to deal with the term

$$R_{M+3} = \sum_{n=M+3}^{\infty} | b_n | \left\| T_n - \sum_{k=0}^{M} \lambda_k^{(n)} C_k^{(\alpha)} \right\|,$$

for

$$\left\| T_n - \sum_{k=0}^{M} \lambda_k^{(n)} C_k^{(\alpha)} \right\| = \left\| T_n + \sum_{k=0}^{M} | \lambda_k^{(n)} | C_k^{(\alpha)} \right\| \qquad \text{for all} \quad n > M,$$

$$= T_n(1) + \sum_{k=0}^{M} | \lambda_k^{(n)} | C_k^{(\alpha)}(1),$$

since $C_k^{(\alpha)}$ attains its norm at $x = 1$. Now using the fact that the $\lambda_k^{(n)}$ are decreasing in modulus we have

$$R_{M+3} \leqslant 1 + \max\{A_{M+3}, A_{M+4}\} \sum_{n=M+3}^{\infty} | b_n |,$$

where $A_n = \sum_{k=0}^{M} | \lambda_k^{(n)} |$.

The sufficient condition of Theorem 2.1 now becomes

$$\sum_{n=M+3}^{\infty} | b_n | \leqslant \frac{| b_{M+1} | (\lambda_{M+1}^{(M+1)} - 1) + | b_{M+2} | (\lambda_{M+2}^{(M+2)} - 1)}{2 + \max\{A_{M+3}, A_{M+4}\}}.$$

This condition degenerates into the condition for Chebyshev polynomials of the second kind as $\alpha \to 1$, since $A_{M+3}, A_{M+4} \to 0$ as $\alpha \to 1$. Defining

$$Q_{M+i}^{(\alpha)} = \frac{\lambda_{M+i}^{(M+i)} - 1}{2 + \max\{A_{M+3}, A_{M+4}\}}, \qquad i = 1, 2,$$

we have

$$\sum_{n=M+3}^{\infty} | b_n | \leqslant Q_{M+1}^{(\alpha)} | b_{M+1} | + Q_{M+2}^{(\alpha)} | b_{M+2} |.$$

Table I at the end of this paper gives values of $Q_{M+i}^{(\alpha)}$ for varying values of M and α. As in Example I the $Q_{M+1}^{(\alpha)}, Q_{M+2}^{(\alpha)}$ increase with M, for fixed α between zero and one.

TABLE I

	$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$		$\alpha = 1.0$	
M	Q_{M+1}	Q_{M+2}	Q_{M+1}	Q_{M+2}	Q_{M+1}	Q_{M+2}	Q_{M+1}	Q_{M+2}	Q_{M+1}	Q_{M+2}
2	0.126	0.163	0.225	0.305	0.314	0.443	0.404	0.588	0.500	0.750
3	0.160	0.190	0.296	0.362	0.428	0.541	0.574	0.746	0.760	1.000
4	0.187	0.211	0.353	0.410	0.525	0.627	0.728	0.890	1.000	1.250
5	0.208	0.229	0.400	0.451	0.608	0.701	0.869	1.230	1.250	1.500
10	0.279	0.291	0.562	0.594	0.907	0.972	1.433	1.556	2.500	2.750
15	0.321	0.330	0.659	0.684	1.098	1.149	1.840	1.945	3.750	4.000
18	0.341	0.348	0.703	0.725	1.185	1.231	2.037	2.133	4.500	4.750

3. Ultraspherical Results for Large M

In this section we compare the performance of a general ultraspherical expansion for $\alpha > 0$, with that of a Chebyshev series. We cannot expect the Chebyshev series to perform better for every function in $C[-1, 1]$ and for every M. Indeed, if f is defined on $[-1, 1]$ by

$$f = \sum_{n=0}^{4} a_n T_n - T_5 - T_6 + 7T_7,$$

then the Chebyshev expansion of the second kind of degree four is a better approximation to f than the expansion of the first kind, with the same degree. In fact

$$\sum_{n=5}^{7} a_n T_n = -T_5 - T_6 + 7T_7,$$

$$\sum_{n=5}^{7} b_n U_n = -4U_5 - \tfrac{1}{2}U_6 + \tfrac{7}{2}U_7$$

$$= -24C_5^{(1)} - \tfrac{7}{2}C_6^{(1)} + 28C_7^{(1)} = \sum_{n=5}^{7} \bar{b}_n C_n^{(1)},$$

where $C_n^{(1)} = U_n/\| U_n \| = U_n/U_n(1)$.

Note that although Handscomb's result holds (as indeed it must) quite strongly, namely:

$$\sum_{5}^{7} | a_n | = 9 < 55.5 = \sum_{5}^{7} | \bar{b}_n |.$$

This does not imply $\|f - \sum_{n=0}^{4} a_n T_n\| < \|f - \sum_{n=0}^{4} b_n U_n\|$. In fact, using a numerical maximization technique we have

$$\left\|f - \sum_{n=0}^{4} a_n T_n\right\| = 8.14 > 7.89 = \left\|f - \sum_{n=0}^{4} b_n U_n\right\|.$$

However, as the expansions are truncated after higher order terms (specifically after T_5) we can use the theorem of Rivlin and Wilson [7] to show that

$$\left\|f - \sum_{n=0}^{M} a_n T_n\right\| \leqslant \left\|f - \sum_{n=0}^{M} b_n U_n\right\|, \qquad M = 5, 6,\dots .$$

From this, and other similar examples we formulate:

Conjecture. Suppose $f = \sum_{n=0}^{\infty} a_n T_n = \sum_{n=0}^{\infty} b_n^{(\alpha)} C_n^{(\alpha)}$, where both these expansions are uniformly convergent on $[-1, 1]$. Then there exists M_0 such that

$$\left\|f - \sum_{n=0}^{M} a_n T_n\right\| \leqslant \left\|f - \sum_{n=0}^{M} b_n^{(\alpha)} C_n^{(\alpha)}\right\| \qquad \text{for all} \quad M > M_0 .$$

In a limited set of circumstances we can verify this conjecture. From Abramowitz and Stegun [1] we have

$$C_n^{(\alpha)} = \mu_n^{(n)} C_n^{(\alpha+1)} - \mu_{n-2}^{(n)} C_{n-2}^{(\alpha+1)} \qquad \text{for} \quad \alpha > -\tfrac{1}{2}.$$

In this relation we can establish

LEMMA 3.1. *If* $C_n^{(\alpha)} = \mu_n^{(n)} C_n^{(\alpha+1)} - \mu_{n-2}^{(n)} C_{n-2}^{(\alpha+1)}$ *and* $\alpha \geqslant 0$, *then* $\mu_n^{(n)} \to \infty$ *as* $n \to \infty$.

Proof. From [1], recalling the difference in normalization, we have

$$\begin{aligned}\mu_n^{(n)} &= \alpha \binom{n + 2\alpha + 1}{n} (n + \alpha)^{-1} \left[\binom{n + 2\alpha - 1}{n}\right]^{-1} \\ &= (n + 2\alpha + 1)(n + 2\alpha).\end{aligned}$$

THEOREM 3.2. *Suppose f has the uniformly convergent expansion* $f = \sum_{n=0}^{\infty} a_n^{(\alpha)} C_n^{(\alpha)}$. *Further, suppose there exists $K > 0$ and M_0 such that*

$$| a_{M+1}^{(\alpha)} | + | a_{M+2}^{(\alpha)} | > K \sum_{n=M+3}^{\infty} | a_n^{(\alpha)} | \qquad \textit{for all} \quad M > M_0 .$$

Then there exists M_1 such that

$$\left\|f - \sum_{n=0}^{M} a_n^{(\alpha)} C_n^{(\alpha)}\right\| \leqslant \left\|f - \sum_{n=0}^{M} a_n^{(\alpha+1)} C_n^{(\alpha+1)}\right\| \qquad \textit{for all} \quad M > M_1 .$$

Proof. Using Theorem 2.1 a sufficient condition for

$$\left\| f - \sum_{n=0}^{M} a_n^{(\alpha)} C_n^{(\alpha)} \right\| \leqslant \left\| f - \sum_{n=0}^{M} a_n^{(\alpha+1)} C_n^{(\alpha+1)} \right\|$$

$$2 \sum_{n=M+3}^{\infty} | a_n^{(\alpha)} | \leqslant | a_{M+1}^{(\alpha)} | (\mu_{M+1}^{(M+1)} - 1) + | a_{M+2}^{(\alpha)} | (\mu_{M+2}^{(M+2)} - 1).$$

Since $\mu_{M+1}^{(M+1)}, \mu_{M+2}^{(M+2)} \to \infty$ as $n \to \infty$, then there exists M_1 such that

$$\mu_{M+1}^{(M+1)} - 1 > \frac{2}{K} \qquad \text{and} \qquad \mu_{M+2}^{(M+2)} - 1 > \frac{2}{K}$$

for all $M > M_1$. This gives

$$2 \sum_{n=M+3}^{\infty} | a_n^{(\alpha)} | \leqslant | a_{M+1}^{(\alpha)} | (\mu_{M+1}^{(M+1)} - 1) + | a_{M+2}^{(\alpha)} | (\mu_{M+2}^{(M+2)} - 1)$$

for $M > M_1$, which is sufficient to establish the theorem. A similar theorem can be deduced from Example II of Section 2. Using the notation adopted there, we can establish

THEOREM 3.3. *Let f have the Chebyshev expansion $\sum b_n T_n$ where $\sum | b_n | < \infty$. Suppose further there exist $K > 0$ and M_0 such that*

$$| b_{M+1} | + | b_{M+2} | > K \sum_{n=M+3}^{\infty} | b_n | \qquad \textit{for all} \quad M > M_0 .$$

Then there exists M_1 such that

$$\left\| f - \sum_{n=0}^{M} b_n T_n \right\| \leqslant \left\| f - \sum_{n=0}^{M} a_n^{(\alpha)} C_n^{(\alpha)} \right\| \qquad \textit{for all} \quad M > M_1 ,$$

where $0 < \alpha \leqslant 1$.

Proof. As in Theorem 3.2 we shall suppose that $\sum_{n=0}^{\infty} a_n^{(\alpha)} C_n^{(\alpha)}$ is uniformly convergent. In this case a sufficient condition for

$$\left\| f - \sum_{n=0}^{M} b_n T_n \right\| \leqslant \left\| f - \sum_{n=0}^{M} a_n^{(\alpha)} C_n^{(\alpha)} \right\|$$

is

$$\sum_{n=M+3}^{\infty} | b_n | \leqslant Q_{M+1}^{(\alpha)} | b_{M+1} | + Q_{M+2}^{(\alpha)} | b_{M+2} |,$$

where $Q_{M+i}^{(\alpha)}$ is defined in Section 2, for $i = 1, 2$. If we can show that

$Q_{M+i}^{(\alpha)} \to \infty$ as $M \to \infty$ for $i = 1, 2$, then the same argument applied in Theorem 3.2 will yield the desired result. $Q_{M+i}^{(\alpha)}$ is a quotient, the denominator being

$$2 + \max\{A_{M+3}, A_{M+4}\}.$$

Now $A_{M+3} = \sum_{k=0}^{M} | \lambda_k^{(M+3)} |$, and the $\lambda_k^{(n)}$ are continuous functions of α for $0 \leqslant \alpha \leqslant 1$ with $\lambda_k^{(n)}(\alpha) \leqslant 0$. Hence $A_{M+3} = A_{M+3}(\alpha)$, a continuous function of α with $A_{M+3}(0) = A_{M+3}(1) = 0$. Since $[0, 1]$ is compact there exists R such that $| A_{M+3}(\alpha)| \leqslant R$ for $0 \leqslant \alpha \leqslant 1$. The same argument shows that A_{M+4} is bounded in this range, and hence the denominator of $Q_{M+1}^{(\alpha)}$ is bounded for $i = 1, 2$, and $0 \leqslant \alpha \leqslant 1$.

The numerator of $Q_{M+1}^{(\alpha)}$ is $\lambda_{M+1}^{(M+1)} - 1$ where

$$\lambda_{M+1}^{(M+1)} = \frac{(2\alpha + 1)(2\alpha + 2) \cdots (2\alpha + M - 1)(2\alpha + M)}{(\alpha + 1)(\alpha + 2) \cdots (\alpha + M - 1)(\alpha + M)}$$

$$= \frac{\Gamma(M + 2\alpha + 1)}{\Gamma(M + \alpha + 1)} \cdot \frac{\Gamma(\alpha + 2)\, \Gamma(2\alpha + 1)}{\Gamma(2\alpha + 2)\, \Gamma(\alpha + 1)},$$

and from the properties of these gamma functions we have $\lambda_{M+1}^{(M+1)} \to \infty$ (where α remains fixed throughout). Similarly $\lambda_{M+2}^{(M+2)} \to \infty$ and so $Q_{M+i}^{(\alpha)} \to \infty$ as $M \to \infty$ for $i = 1, 2$ and $0 \leqslant \alpha \leqslant 1$.

We now indicate what degree of smoothness is sufficient to give a result similar to Theorem 3.3 for more general values of $\alpha > 0$. Our smoothness condition will take the form of a restriction on the behavior of the Chebyshev coefficients, and provides *one example* of the type of condition which will suffice.

LEMMA 3.4. *Let f have the Chebyshev expansion $f = \sum a_n T_n$ on $[-1, 1]$, where $2^n | a_n | \to A$ as $n \to \infty$. Then there exists $K(\alpha) \in R$, $K > 0$ such that*

$$| a_{M+1}^{(\alpha)} | + | a_{M+2}^{(\alpha)} | > K(\alpha) \sum_{n=M+3}^{\infty} | a_n^{(\alpha)} | . \qquad \text{for all} \quad M > M_\alpha > 0,$$

where $f = \sum a_n^{(\alpha)} C_n^{(\alpha)}$ and $\alpha = 0, 1, 2, 3,\ldots$.

Proof. The case $\alpha = 0$ is the Chebyshev case and so the result follows in this case from the assumption on the behavior of the $a_n = a_n^{(0)}$ as n gets large. We now assert that $2^n | a_n^{(\alpha)} |/n^\alpha \to B$, where B is a constant independent of n. This can be established by induction, the inductive step from α to $\alpha + 1$ being as follows: there exists $N = N(\alpha) > 0$ such that

$$| a_n^{(\alpha+1)} | = \frac{(n + 2\alpha + 1)(n + 2\alpha)}{2(2\alpha + 1)(n + \alpha)} | a_n^{(\alpha)} | - \frac{(n + 2)(n + 1)}{2(2\alpha + 1)(n + \alpha + 2)} | a_{n+2}^{(\alpha)} |$$

$$\text{for} \quad n \geqslant N.$$

Hence

$$\frac{2^n}{n^{\alpha+1}} | a_n^{(\alpha+1)} | \to \frac{2^n}{n^{\alpha+1}} \cdot \frac{B}{2^{n+1}(2\alpha+1)}$$

$$\times \left[\frac{4(n-2\alpha+1)(n+2\alpha)(n+\alpha+2)\cdot n^\alpha - (n+2)(n+1)(n+\alpha)(n+2)^\alpha}{(n+\alpha)(n+\alpha+2)}\right]$$

$$\to \frac{3B}{2\alpha+1} \quad \text{as} \quad n \to \infty.$$

We start the inductive process from $\alpha = 0$, where we know the result to hold. We now establish the assertion of the lemma. For any *given* $\alpha \in N$ we can find $N(\alpha)$ such that

$$| a_n^{(\alpha)} | < \frac{2Bn^\alpha}{2^n} \text{for all} \quad n > N(\alpha).$$

Hence

$$\sum_{n=M+3}^{\infty} | a_n^{(\alpha)} | < 2B \sum_{n=M+3}^{\infty} \frac{n^\alpha}{2^n}$$

$$\leqslant \frac{2BM^\alpha}{2^{M+3}} + 2B \sum_{r=1}^{\alpha} \binom{\alpha}{r} \frac{M^{\alpha-r}}{2^M} \sum_{p=3}^{\infty} \frac{p^r}{2^p}$$

$$\leqslant \frac{2BM^\alpha}{2^{M+3}} + \frac{2BM^\alpha}{2^M} \sum_{r=1}^{\alpha} \binom{\alpha}{r} \sum_{p=3}^{\infty} \frac{p^r}{2^p}$$

$$\leqslant \frac{2BM^\alpha}{2^{M+3}} \cdot H,$$

where H is a constant *independent* of M.

Thus given $\epsilon > 0$ we can find $K(\alpha)$ such that

$$K(\alpha) \sum_{n=M+3}^{\infty} | a_n^{(\alpha)} | \leqslant K(\alpha) \cdot \frac{2BHM^\alpha}{2^{M+3}} \leqslant \frac{(M+1)^\alpha}{2^{M+1}} + \frac{(M+2)^\alpha}{2^{M+2}} + \epsilon$$

and the proof is complete.

THEOREM 3.5. *Let f have the Chebyshev expansion $f = \sum b_n T_n$ on $[-1, 1]$ where $2^n | b_n | \to A$ as $n \to \infty$. Given any $\alpha > 0$, there exists $N(\alpha)$ such that, in the notation of Lemma* 3.4,

$$\left\| f - \sum_{n=0}^{M} b_n T_n \right\| < \left\| f - \sum_{n=0}^{M} a_n^{(\alpha)} C_n^{(\alpha)} \right\| \qquad \textit{for all} \quad M > N(\alpha).$$

Proof. Using Lemma 3.4 and Theorem 3.3 the result holds for $0 < \alpha \leqslant 1$. If α is an integer greater than one, then the result follows from Lemma 3.4

and Theorem 3.2. If α is not an integer, then we know from Lemma 3.4 that there exists K such that

$$| a_{M+1}^{(\alpha_0)} | + | a_{M+2}^{(\alpha_0)} | > K \sum_{n=M+3}^{\infty} | a_n^{(\alpha_0)} | \quad \text{for} \quad M > M_{\alpha_0},$$

where α_0 is the integer part of α. We will be able to repeat the argument of Theorem 3.3 in this case if we can show that in the representation[1]

$$C_n^{(\alpha_0)} = \sum_{r=0}^{[n/2]} \lambda_r^{(n)} C_{n-2r}^{(\alpha)}$$

the coefficients $b_r^{(n)}$ satisfy $| \lambda_{r+1}^{(n+2)} | < | \lambda_r^{(n)} |$ and $\lambda_r^{(n)} < 0$ for $| \leqslant r < [n/2]$. A formula for the $\lambda_r^{(n)}$, due to Gegenbauer and given in [2], is

$$\lambda_r^{(n)} = \frac{\Gamma(\alpha)(n - 2r + \alpha)\, \Gamma(r + \alpha_0 - \alpha)\, \Gamma(n - r + \alpha_0)}{\Gamma(\alpha_0)\, \Gamma(\alpha_0 - \alpha)\, \Gamma(n - r + \alpha + 1)\, r!}.$$

The fact that $\lambda_r^{(n)} < 0$ for $1 \leqslant r \leqslant [n/2]$ follows immediately from the observation that all the arguments to the gamma functions are positive, except for $\Gamma(\alpha_0 - \alpha)$ which is negative for $\alpha \in (\alpha_0, \alpha_0 + 1)$. The quotient $| \lambda_{r+1}^{(n+1)} |/| \lambda_r^{(n} | = \lambda_{r+1}^{(n+2)}/\lambda_r^{(n)}$ for $1 \leqslant r \leqslant [n/2]$ may be evaluated as

$$\frac{\lambda_{r+1}^{(n+2)}}{\lambda_r^{(n)}} = \frac{(r + \alpha_0 - \alpha)(n - r + \alpha_0)}{(r + 1)(n - r + \alpha + 1)}$$

which is less than unity since $1 \leqslant r \leqslant [n/2]$ and $0 < \alpha - \alpha_0 < 1$. Last, we need to show that $\lambda_0^{(n)} \to \infty$ as $n \to \infty$. The form of $\lambda_0^{(n)}$ is

$$\begin{aligned} \lambda_0^{(n)} &= \frac{\Gamma(\alpha)(n + \alpha)\, \Gamma(n + \alpha_0)}{\Gamma(\alpha_0)\, \Gamma(n + \alpha + 1)} \\ &= \frac{\Gamma(\alpha)\, \Gamma(n + \alpha_0)}{\Gamma(\alpha_0)\, \Gamma(n + \alpha)} \to \infty \quad \text{as} \quad n \to \infty. \end{aligned}$$

We now have sufficient information about the $\lambda_k^{(n)}$ to ensure that the arguments used in the proof of Theorem 3.3 may be carried over to the situation we have here.

4. Remarks

The Theorem 2.1 is not a particularly deep result, and yet it has yielded some quite powerful statements in Section three. Although Theorem 3.3

[1] Here we have for convenience abandoned temporarily the normalization $C_n^{(\alpha)}(1) = 1$ and used instead $C_n^{(\alpha)}(1) = \binom{n+2\alpha-1}{n}$.

does not cover a wide range of values for α (in fact, $0 < \alpha \leqslant 1$), it does include the Legendre polynomials and the Chebyshev polynomials of the second kind. Thus any function whose Chebyshev coefficients satisfy

$$| b_{M+1} | + | b_{M+2} | > K \sum_{n=M+3}^{\infty} | b_n | \tag{1}$$

for all $M > M_0$, will have the property that the error in approximating the function by truncated Legendre expansions or Chebyshev expansions of the second kind will eventually, for sufficiently large M, have uniform error greater than that of the Chebyshev expansion with the same number of terms. The result for general $\alpha > 0$ is harder to obtain since the only possible approach seems to be that adopted by Lemma 3.4 and Theorem 3.5, where we ensured that property (1) was inherited by successive ultraspherical expansions for $\alpha = 1, 2, 3, \ldots$. It is worth noting that Rivlin and Wilson's result (outlined in Section 1) is a consequence of Handscomb's results in the case $\alpha = 0$, and both authors in effect exploit the same property of the ultraspherical polynomials, namely for $\alpha_1 > \alpha \geqslant 0$

$$C_n^{(\alpha_1)} = \sum_{r=0}^{n} e_r^{(n)} C_r^{(\alpha)},$$

where $e_r^{(n)} \geqslant 0$. In order to obtain conditions on the *Chebyshev* coefficients we have exploited in this paper the representation of the $C_n^{(\alpha)}$ in terms of the $C_r^{(\alpha_1)}$ for $0 \leqslant r \leqslant n$.

Finally, Lemma 3.4 and Theorem 3.5 are only examples of the type of result that can be achieved by applying Theorem 2.1. Both of these results would remain valid if the hypothesis on the Chebyshev coefficients was replaced by $0 < \epsilon < | a_n | \lambda^n < A$ for $\lambda > 1$.

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